Advanced Math: Notes on Lessons 98-101

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Lesson 98: Change of Base / Contrived Log Problems

Change of Base

Many calculators can only compute the log for base 10 or e. That's not a problem; for an arbitrary base b, you can compute the log this way:

$$\log_b x = \frac{\log_{10} x}{\log_{10} b} \textit{ where } b \neq 0, x > 0$$

You can use any base (such as e) instead of base 10, as long as you're consistent (e.g., use e instead of 10 in both places). Saxon thinks this is "hard to remember", but I've never found it hard to remember. Just take the log, and divide it by the log of the desired base. Saxon suggests that you derive this each time; that's a pain, frankly. If you have to re-derive it, derive it in a general form (like above) so you can reuse it! Here's how to derive it:

$y = \log_b x$	To solve; $b \neq 0$ and $x > 0$
$b^y = x$	Exponentiate both sides
$\log_{10} b^y = \log_{10} x$	Take log of both sides
$y \log_{10} b = \log_{10} x$	Move exponent out
$y = \log_b x = \dfrac{\log_{10} x}{\log_{10} b}$	Solve for y

So for example, you can find $\log_2 16 = \log_{10} 16 / \log_{10} 2 = 1.204119983../0.301029996.. = 4$. If you do this on a typical calculator, there will be a small error due to roundoff. Nearly all logarithms are irrational numbers, so most calculators can only approximate their true value.

Contrived Log problems

This section has contrived problems to give you practice with logs, in particular:

1. $\log x^n = n \log x$
2. You can often solve exponential equations (where the unknown is in the exponent) by taking the log of both sides
3. You can often solve logarithmic equations (where the unknown is the parameter of log) by exponentiating both sides (by the base of the log)
4. $\log x + \log y = \log xy$
5. $\log x - \log y = \log (x/y)$

As with the trig identities, *practice* the problems – there's no substitute for doing these yourself.

Lesson 99: Sequence Notations / Advanced Sequence Problems / Arithmetic and Geometric Means

The n^{th} arithmetic term is $a_n = a_1 + (n-1)d$

The n^{th} geometric term is $a_n = a_1 r^{n-1}$

The "n-1" term is because the first term is set separately.

When someone asks for "the arithmetic mean of two values", they mean the single mean between them – i.e., the average.

When someone asks for the "geometric mean of two values", they mean the single term between two other values in a geometric sequence. This one is harder to remember, so you'll typically need to derive it. If the two known terms are x and y, then we have a geometric sequence of 3 terms: x, geometric mean, and y. This means that $y = a_3 = a_1 r^{n-1} = xr^{3-1} = xr^2$. We can now solve for r:

$$y = x r^2 \text{implies } r = \sqrt{y/x}$$

But now that we know the common multiplier, and we know the first term (x), we can solve for the second term:

$$\text{geometric mean} = x\sqrt{y/x} = \pm\sqrt{xy}$$

Lesson 100: Product Identities / More Sum and Difference Identities

Product Identities

There are more identities for the product of a cosine or sine of one angle with the cosine or sine of another angle, such as cos A sin B. You can derive any of them from the identities for sin (A±B) and cos (A±B); write down those identities, and combine the ones that have the identity you want.

First, here are the identities for sin (A±B) and cos (A±B):

- sin (A + B) = sin A cos B + cos A sin B
- sin (A - B) = sin A cos B - cos A sin B
- cos (A + B) = cos A cos B - sin A sin B
- cos (A - B) = cos A cos B + sin A sin B

Now you can find the identity of cos A sin B; first notice that the first two identities have cos A sin B, so those are the ones you need to use. If you just added them together they'd cancel out, so negate the second one and add, resulting in:

sin (A + B) - sin (A - B) = 2 cos A sin B

cos A sin B = ½(sin (A + B) - sin (A - B))

Sum/Difference Identities

There are identities for adding and finding the difference of two sines, and two cosines. They're complicated, unfortunately. You can derive these from sin (A±B) and cos (A±B) by creating new variables x and y, where x=A+B and y=A-B. You should be able to see that, given those definitions, A=½(x+y) and B=½(x-y). Now since:

- sin (A + B) = sin A cos B + cos A sin B
- sin (A - B) = sin A cos B - cos A sin B

Replacing A+B, etc., produces:

- sin x = sin ½(x+y) cos ½(x-y) + cos ½(x+y) sin ½(x-y)
- sin y = sin ½(x+y) cos ½(x-y) - cos ½(x+y) sin ½(x-y)

To find out what sin x + sin y equals, just add them up:

- sin x + sin y = 2 sin ½(x+y) cos ½(x-y)

Lesson 101: Zero determinants / 3x3 determinants / Determinant solutions of 3x3 systems / independent equations

A "determinant" is a value you can compute for any square matrix. For example, given this matrix:

$$\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$$

The determinant is:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + dhc + gbf) - (gec + hfa + idb)$$

Basically, to calculate the determinant, add up NW-to-SE from the left, and subtract SW-to-NE from the bottom (wrapping as necessary). You can do them in any order, as long as you do them all.

If you have a set of linear equations, you can use "Cramer's rule" to solve the unknowns, which finds them by dividing one determinant by another. For example, to solve this 3-equation, 3-unknown set of linear equations:

$$\begin{aligned} a_1 x + b_1 y + c_1 z = k_1 \\ a_2 x + b_2 y + c_2 z = k_2 \\ a_3 x + b_3 y + c_3 z = k_3 \end{aligned}$$

The solutions are, for the 3x3 case, as follows (note that the denominators are the same):

$$x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} k_1 & a_1 & c_1 \\ k_2 & a_2 & c_2 \\ k_3 & a_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} k_1 & a_1 & b_1 \\ k_2 & a_2 & b_2 \\ k_3 & a_3 & b_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

"Cramer's rule" is the observation that there's a simple pattern for solving sets of linear equations that works for any number of unknowns. All the solutions are calculated by dividing one determinant by another. The determinant in the denominator is the same for all, and is simply all the coefficients of the variables. If this determinant is zero, you can't solve the equation by variable elimination - because you can't divide by zero. This attempt to divide by zero means that there are no solutions *or* there are an infinite number of solutions. The numerators differ for each unknown, but the left-hand-column are the constants, followed by all the coefficients *except* the coefficients for the variable you're solving.

So let's work through an example; let's solve this system of linear equations:

$$2x+3y+4z=3$$
$$1x+2y+3z=4$$
$$5x+1y+2z=5$$

Note that this has this form:

$$a_1x+b_1y+c_1z=k_1$$
$$a_2x+b_2y+c_2z=k_2$$
$$a_3x+b_3y+c_3z=k_3$$

Compute the denominator first; it's the same each time and you'll need to reuse it, so solving it first saves time. And if it's zero, you'll know you don't have a single solution. So here's that denominator:

$$denominator=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 5 & 1 & 2 \end{vmatrix}=\begin{matrix}(2\times2\times2+1\times1\times4+5\times3\times3) \\ -(5\times2\times4+1\times3\times2+2\times1\times3)\end{matrix} = \begin{matrix}8+4+45 \\ -40-6-6\end{matrix} = 5$$

Now solve for each variable:

$$x=\frac{\begin{vmatrix} 3 & 3 & 4 \\ 4 & 2 & 3 \\ 5 & 1 & 2 \end{vmatrix}}{denominator}=\frac{\begin{matrix}(12+16+45 \\ -40-9-24)\end{matrix}}{5}=0 \qquad y=\frac{\begin{vmatrix} 3 & 2 & 4 \\ 4 & 1 & 3 \\ 5 & 5 & 2 \end{vmatrix}}{denominator}=\frac{\begin{matrix}(6+80+30 \\ -20-45-16)\end{matrix}}{5}=-7$$

$$z=\frac{\begin{vmatrix} 3 & 2 & 3 \\ 4 & 1 & 2 \\ 5 & 5 & 1 \end{vmatrix}}{denominator}=\frac{\begin{matrix}(3+60+20 \\ -15-30-8)\end{matrix}}{5}=6$$

So it appears that x=0,y=-7,z=6, but don't accept that until you've checked your work by plugging the claimed answers into the original equations:

$2(0)+3(-7)+4(6)=3$ correct
$1(0)+2(-7)+3(6)=4$ correct
$5(0)+1(-7)+2(6)=5$ correct

They're all correct, so we have the correct answer to the original problem.