# Advanced Math: Notes on Lessons 58-61
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### Lesson 58: Distance from a point to a line / Narrow and wide parabolas

The distance from a point to a line is defined to be the length of a second line segment that is perpendicular to the line, from the line to the segment (if the point is *on* the original line, the distance is 0). This is extremely important; simulations that involve real-world physics (or pseudo-real-world), such as military simulations and 3D video games, often have to determine if some moving object intercepts another, and they typically do this by computing this distance to see if it's "close enough".

Here's one way to do this:

1. Find the equation of the second (perpendicular) line. This is easy; the new slope is -1/(old slope), and then use the point's position to find b.
2. Find the x and y values the two lines intersect. You have two equations (the old line and new one), solve for the values that satisfy both.
3. Now use the distance formula between that x and y, and the original point.

[ Walk through example 58.1: Find the distance between (-3,5) and y=0.5x+1 ]

The parabola text is really just a small extension of lesson 54.

The general equation of a parabola that opens straight up or down adds a "width" parameter:

$$y=a(x-h)^2+k$$

Where:
h : the axis of symmetry (the value of x where the parabola changes direction; minimum or maximum)
k: the y-value at the axis of symmetry. Positive values move it up.
a: second-order term. If +, the parabola opens up; if -, it opens down.

If |a| is 1, it's normal width. If |a|>1, it's "narrower" because its values get larger or smaller much more quickly as you move away from the axis of symmetry If |a|<1, it's "wider" because its values get larger and smaller less quickly.

Again, if you want to find a parabola for a particular situation, you can use that equation to generate it. If you're given a (second-order) polynomial and want to understand it, rewrite it in this form and it's easier to understand.

### Lesson 59: Advanced Logarithm Problems

This lesson that repeats a concept you already know. Basically, logarithms and exponentiation are opposites, so:

$$b^{\log_b x}=x$$

So when you're given an equation like this:

$$\log_2(x+5)=6$$

Just apply "$2^{\cdots}$" to both sides, resulting in:

$$x+5=2^6$$

which you should be able to solve (x=59).

**Lesson 60: Factorable Trig Equations / Loss of Solutions Caused by Division**

(Saxon's examples are so good in this lesson that I'll quote them here.)

You can factor equations with trig functions; in fact, when you can it's a good idea. First, let's cover how to factor them. Remember that you can factor simple quadratic equations:

$x^2 - 1 = 0$ and $x^2 + x = 0$

into:

$(x+1)(x-1)=0$ and $x(x+1)=0$

The "x" can be replaced with something else, such as "cos x" or "sin x" and it's still true. So:

$\sin^2 x - 1 = 0$ and $\cos^2 x + \cos x = 0$

factors into:

(sin x + 1)(sin x - 1) = 0 and (cos x)(cox x + 1) = 0

You should factor where you can, and avoid using division. The problem with division is that it can "hide" some solutions to equations. This is not unique to trig - division has this problem in general - it's just that it's particularly likely to bite when trig functions are involved.

Here's the example from Saxon, and this is a good one. If you solve:

$2\sin x \cos x = \sin x \qquad 0° \le x < 360°$

by dividing both sides by "sin x", you'll get "2 cos x =1", which has solutions x={60°,300°}. But this is not the full set of solutions; the full set of solutions is actually {0°, 60°, 180°, 300°}. What went wrong?

Saxon notes that division causes problems, but not why. The key problem is division by zero. Division implicitly presumes that the divisor is not 0, but trig functions arrive at 0 quite often; certainly "sin x" does, at 0° and 180°. And unfortunately, not only are those possible values of the numerator, it turns out that they are solutions to the original equation, so the full solution is x={0°, 60°, 180°, 300°}.

This division-by-zero issue is, by the way, a general problem with symbolic calculation systems. Most symbolic math programs completely ignore division-by-zero, and thus omit these solutions entirely. For example, the generally very good (and free!) symbolic math program Maxima (and its graphical front-end wxMaxima) gleefully uses division and arc-trig functions, so it will only find x={60°,300°} and not the full set if you ask it to directly solve the original equation.

The solution is to factor instead, like this:

$$2\sin x \cos x = \sin x \qquad 0° \le x < 360°$$
$$2\sin x \cos x - \sin x = 0$$
$$(\sin x)(2\cos x - 1) = 0$$
$$\sin x = 0 \quad \text{or} \quad 2\cos x - 1 = 0$$

Now it's much easier to see the solutions: x={0°, 60°, 180°, 300°}.

**Lesson 61: Single-Variable Analysis (inc. Standard Deviation) / Normal Distribution / Box-and-Whisker Plots**

The "average" or "mean" of a set of values is simply the sum of the values, divided by the number of values. This can expressed this way:

$$average = \mu = \frac{1}{n}\sum_{i=1}^{i=n} x_i$$

So the average of 3,5,10 is (1/3)(3+5+10) = (1/3)(18) = 6.

The "variance" is simply the "sum of the squares of difference from the mean, divided by the number of values" . This can be expressed this way:

$$Variance = \frac{1}{n}\sum_{i=1}^{i=n} (x_i - \mu)^2$$

So the variance of 3, 5, 10 is (1/3) $((3-6)^2 + (5-6)^2 + (10-6)^2)$ = (1/3)(9+1+16)= 26/3.

The "standard deviation" is just the square root of the variance. In our running example, this is sqrt(26/3) which is approximately 2.94.

Note: While is strictly true, it's not the whole story. Saxon omits an important complication that you should be aware of. If you only have a sample of numbers, instead of the whole set of numbers, you should divide by "n-1" instead of "n" to calculate the variance or standard deviation. Thus, many programs have both a "standard deviation" and a "sample standard deviation" function, where the former divides by "n" and the latter divides by "n-1". I think you should know that now, because otherwise using programs to do this may be confusing to you.

When you collect sets of data, you can graph the frequency of different values to see which values are more common. Often this graph will start to approximate a particularly common curve called the "normal distribution". See lesson 61's graphs for more information. If a distribution is the normal distribution, then 68% of the values will be within one standard deviation of the average value, about 95% of the values will be within two standard deviations, and about 99% of the values will be within three standard deviations.

Remember that the "median" is the central value in the sorted list of values (if there is an even number of values, it's the average of the two center values). The "mode" is the value (or group of values) that is most common.

Box-and-whisker diagrams (aka "box plots") were invented in 1977 by John Tukey, and are really very simple. Sort your data, and group them into 4 equally-sized groups ("quarters"). The "box" surrounds the middle two quarters; then draw a line between the two quarters (the median). Now add "whiskers": lines that show the full range of values, to the smallest and largest. Sometimes people declare some especially large/small points as "outliers", but that can be misleading. Often people mark the mean (average), though that's not noted by Saxon.

Box-and-whisker diagrams can help you understand a large set of data. E.G., on a graph showing stock prices over a period of time, each day might be shown with a box-and-arrow diagram so that you can the range of prices over each day, and then you can see many days' worth of charts. Unfortunately, Microsoft Excel and OpenOffice.org don't build them in; you can make them do it through some complicated maneuvers (e.g., http://support.microsoft.com/kb/155130). The "R" statistical package has this plot type built in.

Below is an example of a box-and-whisker chart. This example shows the data from Michelson and Morley's famous and important experiment trying to measure the speed of light (they made careful measurements and showed that direction did not make a significant difference in its speed, disproving that there was an "ether" with the properties that had been assumed at the time). They performed five experiments, where each made 20 consecutive runs. (Credit: The graphic was originally created by User:Schutz for Wikipedia on 28 December 2006, using the R statistical project, and was downloaded from http://en.wikipedia.org/wiki/Image:Michelsonmorley-boxplot.svg. The graphic was released to the public domain.).