Advanced Math: Notes on Lessons 54-57
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Lesson 54: Parabolas

The general equation of a normal-width parabola that opens straight up or down is this:
$y = (sign)(x-h)^2 + k$
Where:
h : the axis of symmetry (the value of x where the parabola changes direction; minimum or maximum)
k: the y-value at the axis of symmetry. Positive values move it up.
sign: + or -. If +, the parabola opens up; if -, it opens down. (We'll broaden this to any number.)

For some reason Saxon forgets to note "sign" in the general equation, though they discuss it.

So if you want a parabola that opens up, whose axis of symmetry is at x=2, and at that point, you want its y-value to be 3, that's simply:
$y = +(x-2)^2 + 3$

If you're given an equation of the form y=f(x), where x is a polynomial with largest exponent 2, you probably have a parabola of this form. If you factor it into the general form above, it'll be much easier to graph and understand.

Lesson 55: Circular Permutations / Distinguishable Permutations

If you put things in a circle *and* there is no distinguished position in the circle (like a "head chair"), then there aren't as many possible orderings. That's because ABC and BCA are the "same order" when you put them in an (undistinguished) circle. These are called circular permutations. In a circular permutation, all "first" positions are identical, so there's only one first position... then the rest of the positions are normal: n-1, n-2, and so on.

Recall that for n objects, there are n! normal (linear) permutations.
For n objects, there are (n-1)! circular permutations (because the first "n" became "1").

Circular permutations are uncommon, but *distinguishable* permutations happen all the time. So far, we've assumed that every one of the n objects is different (distinguishable), but this is often not true. What do you do when some of the objects are "the same"? E.G., instead of ABC you have MOO (where the Os are not distinguishable)? Or chess pieces, where the two knights of the same color are interchangeable?

The answer is that you determine how many permutations are actually the same (for your purposes). Then you can compute the number of distinguishable permutations = (permutations) / (permutations that are the same).

So for MOO, we have 3 letters. If they were distinguishable, you'd have 3! permutations. But the O's are the same, so there are 2! permutations of Os (which are the same). The distinguishable permutations are (3!)/(2!) = 3. [The distinguishable permutations are MOO, OMO, and OOM.]

When there are multiple groups that aren't distinguishable, you multiply each group's permutations together to get the "permutations that are the same". So if there are 7 marbles, 3 red and 4 green, the number of distinguishable permutations = $7! / (3! \cdot 4!) = (5\cdot6\cdot7)/(1\cdot2\cdot3) = (5\cdot7) = 35$.

Longer example: Grandmaster Bobby Fischer created a chess variant where there are many possible starting positions (typically picked at random). Given the method below for creating a starting position, how many different starting positions are possible? (Note: the back row of a chess board has 8 squares, alternating between 4 white and 4 black squares.)
A method for creating a starting position:

- Place a bishop randomly on one of the 4 white squares
- Place a bishop randomly on one of the 4 black squares
- Place a queen randomly on one of the 6 remaining squares
- Place a knight randomly on one of the 5 remaining squares
- Place the other knight randomly on one of the 4 remaining squares
- There are 3 remaining squares: Fill them with rook, king, rook in that order. (In the real game, you then place all pawns in their usual squares, and Black's pieces mirror White's).

So, how many possible positions are there? If the knights were distinguishable, then we would have this many possible positions:

4 x 4 x 6 x 5 x 4 (x 1 for the rook-king-rook)

But the two knights are not distinguishable; if you swapped them, there'd be no difference. So we must divide by the permutations of the knights = 2! = 2. So the number of possible distinguishable positions is:

4 x 4 x 6 x 5 x 4 (x 1) / (2) = 960 possible initial positions

Lesson 56: TriangularAreas / Areas of Segments / Systems of Inequalities

Area of a triangle is simple (1/2) (base) (height). You already know that; but now that you know trig, you can compute heights that you couldn't before. That's all.

A sector of a circle is just the area bounded by two radii and an arc. The area of a sector (e.g., a piece of pie) is $(\text{angle}/360)\pi r^2$. The segment of an arc is the sector, minus the triangle defined by the arc endpoints and the circle's center. The area of segment is found by just using that definition - find the area of the circle, and subtract the area of the triangle.

You've already seen systems of inequalities with lines, e.g., $y > 2x-1$ with $y < 3x+5$. Basically, you can do this with any inequality, not just lines; they work the same way.

Lesson 57: Phase Shifts in Sinusoids / Period of a Sinusoid

The sine and cosine's shape is a "sinusoid" (the adjective is "sinusoidal"). In fact, the cosine is exactly the same shape as the sine, but shifted by 90°. You can shift a sine or cosine by an arbitrary amount left or right by rewriting it as $\sin(\theta - p)$, where p is the phase angle or phase shift. It's just the amount moved left or right, but because the values of a sine or cosine are angles, that means the shift has to be an angle measure too.

Because cos θ = sin (θ+90°), that means that cos θ is simply the sine, phase-shifted by -90°.

Normally sine and cosine repeat every 360°, but you can make them go through a full period (cycle) more or less rapidly by multiplying their parameter by some coefficient, e.g., sin kθ. The coefficient "k" determines the new period's length:

$$k = \frac{360}{period\ degrees} = \frac{2\pi}{period\ radians}$$

You can combine phase shift and changing the period, too.