Advanced Math: Notes on Lessons 25-28
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[Show off slide rules – which are based on logarithms]

Lesson 25: Age Problems/Rate Problems
This one is straightforward; it shows two common types of word problem and how to solve them. Like any word problems, it's important to first identify some unknowns. Assign them letters or subscripts, and *write down* what each one means so that you and later readers will know what you mean. Then write down the equations; it's best to start by writing the equations as close as possible to their original form (e.g., where "is" and "are" translate to "="), and *then* later adjust them.

Age problems look like "Five years ago, Alice was three times as old as Bob...". You typically assign a symbol (e.g., letter) for each person, to represent their age. The key question is, age in *what year?* It's often useful to make the variable be the age "now" - but whatever you choose, document it! It's not a bad idea to then build up expressions in steps, like this:
A = Alice's age today
B = Bob's age today
A – 5 = Alice's age 5 years ago
B – 5 = Bob's age 5 years ago
Now you can easily represent the first sentence: "Five years ago, Alice was three times as old as Bob":

$$(A-5)=3\times(B-5)$$

Once written, you can then rewrite it as A = 3B – 10; but only do that after writing the first one down.
You could represent "Twenty years from now, Alice will be ten more than Bob's age", by noting:
A + 20 = Alice's age 20 years from now
B + 20 = Bob's age 20 years from now
And then you can represent the whole thing easily as:

$$(A+20)=(B+20)+10$$

Now we can use algebra. Substituting the first definition of "A" yields:

$$(3B-10)+20=B+30 \rightarrow 2B+10=30 \rightarrow 2B=20 \rightarrow B=10$$
$$A=3B-10 = 3(10)-10 = 20$$

So we have B=10 and A=20. But note that these are their *current* ages (we know this, because we carefully wrote that down at the beginning). If the question asks, "what will Bob's age be 20 years from now" (perhaps not exactly in those words), then the answer would be 10+20=30, not 10!! In general, be careful that you answer what's asked. However, that's *especially* important in age problems, since there are many timeframes that the question might ask about & it often isn't "now".

Rate problems are all about something being "accomplished" at a certain fixed rate. The "accomplishment" might be a distance traveled, the volume of water being put in a pool, the number of widgets made, the number of cars washed, the number of pies eaten, and so on. The rate is simply how much is accomplished, divided by the time to accomplish it. If people can work independently (e.g., they can make widgets separately), they may have different rates for accomplishments. E.G., if Alice can make 20 widgets in 2 hours, that's 20/2 = 10 widgets/hour. If Bob can make 28 widgets in 2 hours, that's 28/2 = 14 widgets/hour.

Now let's say that the goal is to perform an accomplishment, e.g., make widgets. The number of accomplishments that can performed at a fixed rate by one performer is simply the rate of that performer times the time that performer does it, that is:

Accomplishments = Rate x time

So if Bob can make 14 widgets/hour, and Bob has 8 hours, that means Bob can make 14x8=112 widgets in that time.

If people are all working to perform a task, but they work strictly independently, you can add up their accomplishments separately, and the total is how much they do together. So if Bob works for 8 hours, and Alice works for 9 hours, at their respective rates together they can make this many widgets:

Accomplishments = (Bob's accomplishments in 8 hours) + (Alice's accomplishments in 9 hours)

= 14x8 + 10x9 = 112 + 90 = 202 widgets.

If all the performers (workers) have the same rate, and do it for the same amount of time, you'd have:

Accomplishments = (Worker 1's accomplishments) + (Worker 2's) + (Worker 3's) + …

= rate x time + rate x time + rate x time + ….

but that's just adding up the same value for "number of workers" times, so it's really just:

= (rate) (number of workers)(time) = RWT

So if 40 workers can on average make 6 widgets/hour, in 7 hours they can produce:

= (6 widgets/worker-hour) (40 workers)(7 hours) = 6 x 40 x 7 = 1680 widgets

Lesson 26: Log Form of Exponential / Logarithmic Equation

What are logarithms? They're a special operation that's an "inverse" of exponentiation. This:

$N = b^L \qquad \text{where } b>0, b\neq 1, n>0$

means the same thing as:

$\log_b N = L \qquad \text{where } b>0, b\neq 1, n>0$

Why do we have logarithms? Well, one reason is that we need an "inverse" to solve some problems. Addition and subtraction are inverses of each other; multiplication and division are inverses of each other. We can already handle *some* exponentiation problems. When you have to-the-power of a constant value:

$n^3=27$

you can use root-of with the same constant value to get the answer:

$\sqrt[3]{n^3} = \sqrt[3]{27} \quad \rightarrow \quad n=3$

But what happens if the unknown is in the power-of (or root-of) position? I.E., how do we solve this?:

$3^n = 19683$

The answer is, "logarithms". Logarithms are an inverse of the power-of operation. To solve this equation, you take the logarithm of both sides:

$\log_3 3^n = \log_3 19683$

$n = 9$

Here are the two basic rules on logs and exponents:

$\log_b b^n = n \text{ and } b^{\log_b n} = n \qquad \text{where } b>0, b\neq 1, n>0$

Slide rules are based on logarithms, and until calculators were invented, logarithms were absolutely vital for solving practical problems.

The value “b” notated above is called the “base”. Common bases are 10, 2, and *e* (2.71828....). Unfortunately, many documents omit the base and say “it’s obvious”, where in fact it’s *not* obvious; this is due to a long history of different disciplines all using logarithms, but ending up with different conventions for the base. On most calculators, log() uses b=10; if you want base *e*, you use the ln() function instead. ISO standard 31-11:1992 (Mathematical signs and symbols for use in physical sciences and technology) suggests these notations:

- ln(x) means $\log_e(x)$, that is, base=e.
- lg(x) means $\log_{10}(x)$, that is, base=10.
- lb(x) means $\log_2(x)$, that is, base=2.

In class, for now just always include the base (b) when writing a logarithm.

Computing “$\log_{10}$ N” when N is 1 followed by zero or more 0s is easy; the answer is just the number of 0s. So $\log_{10} 1=0$, $\log_{10} 10=1$, $\log_{10} 100=2$, $\log_{10} 1000=3$, and so on.

Logarithms are an example of a function that takes two parameters; on a spreadsheet, LOG(3, 10) is likely to produce $\log_{10}3$. Many calculators can’t compute logs to arbitrary bases; we’ll discuss later how to compute them in such cases.

Lesson 27: Related Angles / Signs of Trig Functions
This is straightforward. Remember, by convention positive angles go counterclockwise, so negative angles go clockwise. Adding or subtracting 360° doesn’t change an angle (presuming that we’re only measuring direction, and not the number of rotations to get there); adding or subtracting 180° reverses it.

Lesson 28: Factorial Notation / Abstract Rate Problems
Factorials are easy; n! = n * (n-1) *....*1. So 5! = 5*4*3*2*1. On many computers it’s written in functional notation, e.g., FACT(n).

THIS WEEK: Work through lessons 25-28 this week (“Monday – Thursday”).

Do home study test #6 this week (covers lessons 21-24), presumably on Friday. Please review your notes before taking it, and memorize what you need first. *Show your work*, so can give partial credit. Have a parent grade it initially (to identify which ones are right or wrong), and put it in the new mailbox on Sunday morning. I intend to do the final grading, so that I can give partial credit (or full credit if it’s just an equivalent expression).