Advanced Math: Notes on Lessons 102-105

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Lesson 102: Binomial Expansions

Lesson 77 (page 472) covered binomial expansions. Here we'll do them again, but where the first and/or second terms aren't simple variables. This is no big deal at all; just find the "simple" binomial solution, and then substitute.

For example, to find $(2x^2 - 3y^2)^4$:

First substitute $A=2x^2$, and $B=- 3y^2$. Now find $(A+B)^4$ using the binomial expansion:

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \\ 1\ 4\ 6\ 4\ 1 \end{array}$$

So $(A+B)^4 = A^4 + 4A^3B^1 + 6A^2B^2 + 4A^1B^3 + B^4$

Now substitute $A=2x^2$, and $B=- 3y^2$:

$16x^8 + (4)(8x^6)(-3y^2) + (6)(4x^4)(9y^4) + (4)(2x^2)(-27y^6)+81y^8$

$= 16x^8 - 96x^6y^2 + 216x^4y^4 -216x^2y^6+81y^8$

Lesson 103: Calculations with Logarithms / Power of the Hydrogen

Calculations with Logarithms

This simply continues practicing with logarithms; it doesn't introduce anything new. Remember:

1. $\log x^n = n \log x$
2. You can often solve exponential equations (where the unknown is in the exponent) by taking the log of both sides
3. You can often solve logarithmic equations (where the unknown is the parameter of log) by exponentiating both sides (by the base of the log)
4. $\log x + \log y = \log xy$
5. $\log x – \log y = \log (x/y)$

As with the trig identities, *practice* the problems – there's no substitute for doing these yourself.

Power of the Hydrogen

The "power of the hydrogen", usually abbreviated "pH", is a measure of how acidic or alkaline a solution is. It is approximately the same as a logarithmic measure of the concentration of hydrogen ions (and thus acidity), ranging from about 0 to +14 (neutral is +7). As the pH becomes smaller than 7, the solution is increasingly acidic; as the pH becomes larger than 7, the solution is increasingly alkaline (aka basic). pH is approximately related to the density of hydrogen ions (in moles/liter), labeled H^+, by the following relationship:

$pH = -\log H^+ = \log (1/H^+)$

You can solve for H^+ by (using the two left expressions) negating and applying 10^X to both sides:

$10^{-pH} = H^+$

So if the concentration $H^+ = 2x10^{-5}$ mole/liter, the pH is:

$pH = -\log (2x10^{-5}) = 4.698970... = 4.70$

A mole is a convenient way of counting atoms, molecules, ions, and so on, which tend to be large numbers. One mole is simply 6.0221415×10^{23} items; so if you know you have N H^+ ions, the number of moles of H^+ ions is $N/(6.0221415\times10^{23})$. (This unusual number is selected so that a substance's atomic or molecular mass in atomic mass units is the same as its molar mass in grams. For more, see your Chemistry class!) An ion is an atom/molecule where the number of electrons and protons differ.

Here's a note that Saxon doesn't include. pH was originally defined as being exactly " $-\log H^+$", but it turns out that hydrogen ions tend to interact with other components of the solution and this affects how acidic or alkaline the solution is. Officially, measuring just the hydrogen ions is p[H], and pH adds a correction factor. For purposes of the class, just know that we're using a useful approximation.

Lesson 104: Arithmetic Series / Geometric Series

A "series" is just the sum of the members of a sequence. So an "arithmetic series" is the sum of an arithmetic sequence, and a "geometric series" is the sum of a geometric sequence. Here, you'll learn special formulas for calculating arithmetic series and geometric series; be sure to *understand* how to create these formulas (you can memorize them if you want, but don't *just* memorize them).

Arithmetic Series

An arithmetic sequence is just a sequence of numbers, where each number in the sequence has the same constant added to the previous one. When you create an arithmetic series by adding them together, an interesting pattern emerges. Let's show it by example, where k=3, and the first value is 4:

Sequence: 4, 7, 10, 13, 16, 19

Series: 4 + 7 + 10 + 13 + 16 + 19

Let's add these up "outside in", and you'll notice something interesting (when you have an even number of values): Each pair adds up to the same number (in this case, 23)! The reason should quickly be obvious: starting from the right, each value is "k" less, but starting from the left, each value is "k" more, and when you add each pair up, the difference cancels out (because k-k=0). So if you have n values in the sequence, you will have n/2 pairs, right? So an arithmetic series will have this value:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Saxon only shows that this is true for series with an even number of values, but it's true for odd ones too. When you have an odd number of values, you have a number in the center that isn't paired with another number.... but when you add it to itself, and divide by two, you get the original number back. So, the final equation turns out to be identical for an odd number of values too.

So our series' total is (6/2)(4+19) = 3(23) = 69, which is the same answer if you add them directly.

Geometric Series

You can find a general equation for the geometric series this way:

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-2} + a_1 r^{n-1}$$

Multiply equation by r:

$$r S_n = \qquad r a_1 + a_1 r^2 + a_1 r^3 + a_1 r^4 + \ldots + a_1 r^{n-1} + a_1 r^n$$

Find first - second:

$$S_n - r S_n = a_1 - a_1 r^n$$

$$(1-r) S_n = a_1 (1-r^n)$$

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{a_1(r^n-1)}{r-1}$$

In this last equation, the middle equation is what Saxon shows. That answer is correct, but a pain to calculate with. I've multiplied the top and bottom by -1, which produces a slightly different format that I find easier to calculate with (finding x-1 is easier than finding 1-x).

So if r=2, a_1=3, and we have a sequence of 4 values, the series is 3 + 6 + 12 + 24. Using the equation, the sum is (3)(2^4-1) /(2-1) = 3(15)/1 = 45.

Lesson 105: Cofactors / Expansion by Cofactors

This lesson defines two new terms involving determinants of matrices ("minor" and "cofactor"), and then shows how to use cofactors to compute a determinant.

Cofactors

As noted in lesson 101, the determinant:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + dhc + gbf) - (gec + hfa + idb)$$

You can pick any element in the determinant's matrix, cross out all the elements in the same row and column, and create a new, smaller determinant which is called the *minor* of that element. So given the 3x3 determinant above, the minor of h is:

$$\begin{vmatrix} a & \cancel{b} & c \\ d & \cancel{e} & f \\ \cancel{g} & \cancel{h} & \cancel{i} \end{vmatrix} = \begin{vmatrix} a & c \\ d & f \end{vmatrix} = af - dc$$

The *cofactor* is the minor expression, multiplied (in its entirety) by the "cofactor sign". The cofactor sign is + for the top leftmost element, and alternates with "-" in both directions. For example, in the 3x3 case, it is:

+ - +
- + -
+ - +

As you can see, for "h" the cofactor sign is "-", so the cofactor = -(af-dc) = dc - af.

Expansion by cofactors

You can calculate a determinant using cofactors. Pick a row or a column; then add up the product of each element by their cofactor. Prefer the row/column with the most zeros, it will make calculating easier (since 0 times anything is 0). We could use the second row, for example:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (-)d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + (+)e\begin{vmatrix} a & c \\ g & i \end{vmatrix} + (-)f\begin{vmatrix} a & b \\ g & h \end{vmatrix} = -dbi + dhc + eai - egc - fah + fgb$$

Note that this equals (aei+dhc+gbf) - (gec+hfa+idb), which was how we calculated determinants earlier.